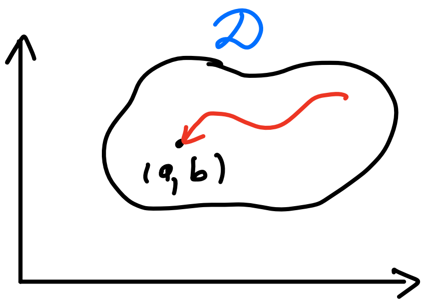
Limits

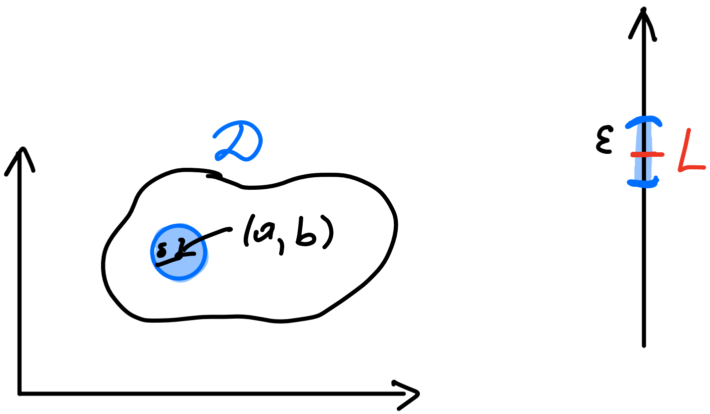
Limits:

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$ – The limit of $f(x,y)$ as (x,y) approaches (a,b)



- if values of $f(x,y)$ approach L as (x,y) approaches (a,b) along any path in the domain $\mathcal{D}$

- we can make $f(x,y)$ as close to L as we want by taking (x,y) sufficiently close to (a,b)



If for every number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ s.t. if $(x,y) \in \mathcal{D}$ and $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$

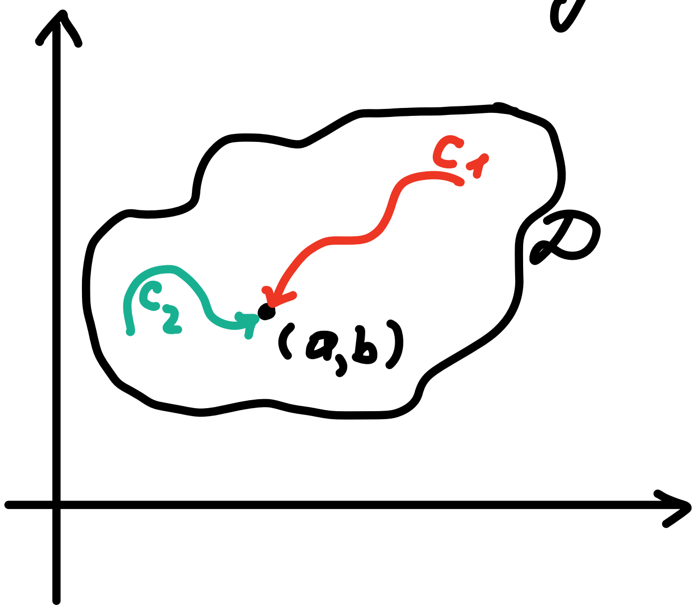
then $|f(x,y) - L| < \varepsilon$

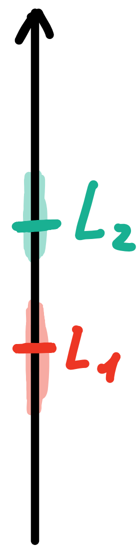
If $f(x,y) \to L_1$ as $(x,y) \to (a,b)$ along C_1

AND $f(x,y) \to L_2$ as $(x,y) \to (a,b)$ along C_2

where $L_1 \neq L_2$, then

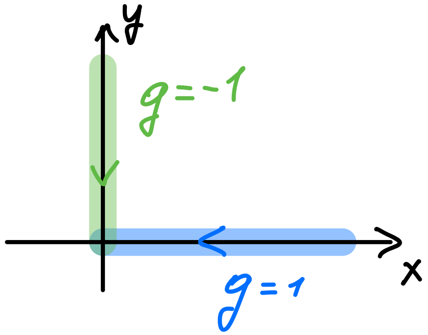
$$\lim_{(x,y)\to(a,b)} f(x,y) \text{ does } \textbf{NOT} \text{ exist.}$$





Ex: $g(x,y) = \dfrac{x^2 - y^2}{x^2 + y^2}$ $\qquad \lim\limits_{(x,y)\to 0} g(x,y) = ?$

Sol:



If $y=0$, we have $g(x,0) = \dfrac{x^2 - 0^2}{x^2 + 0^2} = 1$

approaching along x-axis g stays 1

approaching along y-axis g stays -1

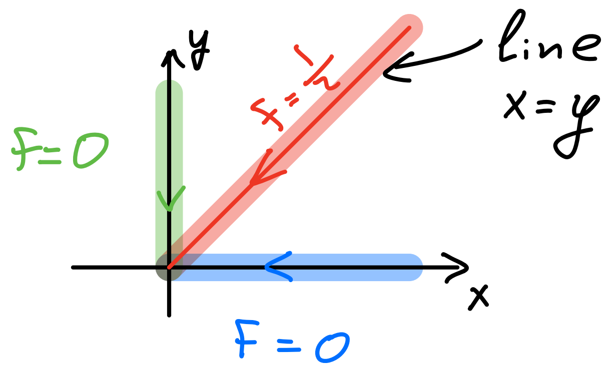
If $x=0$, we have $g(0,y) = \dfrac{0^2 - y^2}{0^2 + y^2} = -1$

$\Rightarrow$ limit as $(x,y) \to (0,0)$ does NOT exist!

Ex: $f(x,y) = \dfrac{xy}{x^2 + y^2}$ $\qquad \lim\limits_{(x,y)\to 0} f(x,y) = ?$

$f(x,0) = 0 = f(0,y)$

BUT $f(x,x) = \dfrac{xx}{x^2 + x^2} = \dfrac{1}{2}$



approaching along x-axis f stays 0

approaching along y-axis f stays 0

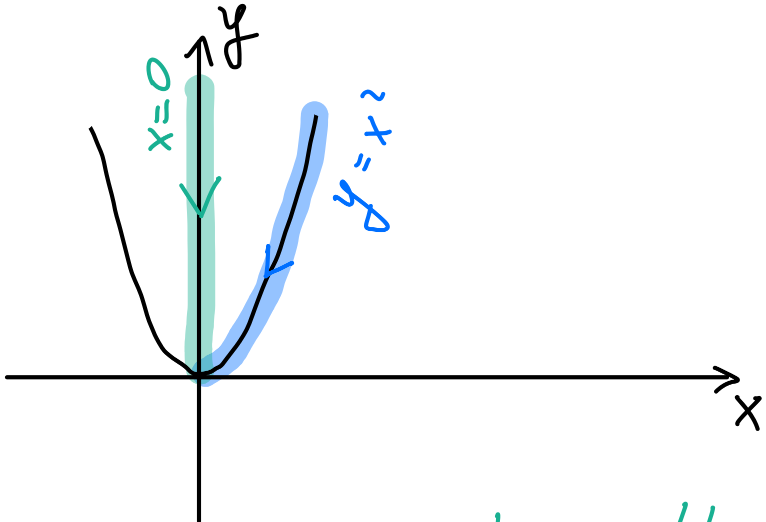
BUT approaching along $x=y$ line $f = \frac{1}{2}$

$0 \neq \frac{1}{2}$

$\Rightarrow$ limit as $(x,y) \to (0,0)$ does NOT exist!

Ex: $f(x,y) = \dfrac{y}{x^2+3y}$ $\qquad \lim\limits_{(x,y)\to(0,0)} f(x,y) = ?$

Sol:



[y-axis is a line $x=0$]

Note that

$$f(0,y) = \frac{y}{0^2+3y} = \frac{y}{3y} = \frac{1}{3}$$

For any $y \in (0,+\infty) \Rightarrow$

Approaching the point $(0,0)$ along y-axis $f = \frac{1}{3}$

Now, let us look at points on the parabola $y = x^2$:

$$f(x, x^2) = \frac{x^2}{x^2+3x^2} = \frac{x^2}{4x^2} = \frac{1}{4} \quad \text{For any } x \in (0,+\infty) \Rightarrow$$

Approaching the point $(0,0)$ along $y=x^2$ line f stays $\frac{1}{4}$

$\frac{1}{3} \neq \frac{1}{4} \Rightarrow$ the limit as $(x,y) \to (0,0)$ does NOT exist

Ex: Find $\lim\limits_{(x,y)\to(0,0)} \dfrac{3x^2y}{x^2+y^2}$ if it exists. I

Let us look at different lines/curves going through the point $(0,0)$ and see if approaching along them we get the same values or not:

$$f(x,y) = \frac{3x^2y}{x^2+y^2}$$

$x=0$: $\quad f(0,y) = 0$

$y=0$: $\quad f(x,0) = 0$

$y=ax$: $\quad f(x,ax) = \dfrac{3x^2(ax)}{x^2+(ax)^2} = \dfrac{3ax^3}{(a^2+1)x^2} = \dfrac{3a}{a^2+1}x \xrightarrow[x\to 0]{} 0$

$y=x^2$: $\quad f(x,x^2) = \dfrac{3x^2x^2}{x^2+(x^2)^2} = \dfrac{3x^4}{x^2+x^4} = \dfrac{3x^2}{1+x^2} \xrightarrow[x\to 0]{} 0$

So we suspect now the limit to exist and $=0$, let us prove it.

Ex: Find $\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2}$ if it exists.

II

Sol: 1) We suspect the limit is 0.

2) Let $\varepsilon > 0$, WANTED $\delta > 0$ such that

if $0 < \sqrt{x^2+y^2} < \delta$ then $\left|\frac{3x^2y}{x^2+y^2} - 0\right| < \varepsilon$

$\left(\Longleftrightarrow \frac{3x^2|y|}{x^2+y^2} < \varepsilon\right)$

3) Note that $x^2 \leq x^2+y^2 \Longrightarrow \frac{x^2}{x^2+y^2} \leq 1$ therefore

$$\frac{3x^2|y|}{x^2+y^2} \leq 3|y| = 3\sqrt{y^2} \leq 3\sqrt{x^2+y^2} \leq 3\delta$$

CHOOSE $\delta = \frac{\varepsilon}{3}$

$$\boxed{\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} = 0}$$

Solution 2:

III

Note that $0 \le \left|\frac{3x^2y}{x^2+y^2}\right| = \frac{3x^2|y|}{x^2+y^2}$.

Note also, that $x^2 \le x^2+y^2$ for any x, y

$\Rightarrow \frac{x^2}{x^2+y^2} \le 1$

Therefore $\frac{3x^2|y|}{x^2+y^2} \le 3|y|$.

So we get $0 \le \left|\frac{3x^2y}{x^2+y^2}\right| \le 3|y|$.

If $h(x,y) \le f(x,y) \le g(x,y)$ for all x, y and $\lim_{(x,y)\to(a,b)} h(x,y) = \lim_{(x,y)\to(a,b)} g(x,y) = L$

then $\lim_{(x,y)\to(a,b)} f(x,y) = L$.

Set $h(x,y) = 0$; $g(x,y) = |y|$. Note that $\lim_{(x,y)\to(0,0)} h(x,y) = \lim_{(x,y)\to(0,0)} g(x,y) = 0$

$\Rightarrow \lim_{(x,y)\to(0,0)} \left|\frac{3x^2y}{x^2+y^2}\right| = 0 \Rightarrow \lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} = 0.$

Continuity

- $f(x,y)$ is continuous at (a,b) if $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$
- f is continuous on $\mathcal{D}$ if f is continuous at each point (a,b) in $\mathcal{D}$

Ex: a polynomial: $f(x,y) = x^2 + xy + y^7$ is cont. on $\mathbb{R}^2$

a rational function: $g(x,y) = \dfrac{x+2y^2}{x^2+y^2}$ is cont. in its domain $\mathbb{R}^2 \setminus \{0,0\}$

Ex: Evaluate $\lim_{(x,y)\to(1,2)} (x^2y^3 - x^3y^2 + 3x + 2y)$

Sol: $f(x,y) = x^2y^3 - x^3y^2 + 3x + 2y$ is a polynomial $\Rightarrow$ cont. on $\mathbb{R}^2$

$$\Rightarrow \lim_{(x,y)\to(1,2)} f(x,y) = f(1,2) = 1^2 2^3 - 1^3 2^2 + 3(1) + 2(2) = 11$$

Ex: Where is $f(x,y) = \dfrac{x^2-y^2}{x^2+y^2}$ continuous?

Sol: Discontinuous at $(0,0)$ [NOT defined there]

Rational fct. is continuous on its domain $\mathcal{D} = \{(x,y) \mid (x,y) \neq (0,0)\}$